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Pfaffianization of the two-dimensional Toda lattice

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Abstract

Pfaffianization procedure due to Hirota and Ohta is applied to the two-dimensional Toda lattice. As a result, a Pfaffianized version of the two-dimensional Toda lattice is found.

In 1991, Hirota and Ohta [1,2] developed an effective procedure of what we now call Pfaffianization, which allows one to produce new coupled systems of equations with solutions in the form of Pfaffians. Recently, there has been growing interest in this direction. For example, such a procedure has been successfully applied to the Davey–Stewartson equations [3], the discrete KP equation [4] and the self-dual Yang–Mills equation [5]. Besides, the Pfaffianized KP hierarchies have been investigated in [6].

The purpose of this paper is to apply the procedure of Hirota and Ohta to the following bilinear two-dimensional Toda lattice:

$$D_x D_s \tau_n \cdot \tau_n = 2(e^{D_n} - 1)\tau_n \cdot \tau_n, \tag{1}$$

where the bilinear operators $D_x D_s$ and $\exp(D_n)$ are defined by [7,8]

$$D_x D_s a \cdot b \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial s'}\right)a(x,s)b(x',s')|_{x'=x,\, s'=s}$$

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and

$$\exp(D_n)a_n \cdot b_n \equiv a_{n+1}b_{n-1}.$$

As a result, the Pfaffianized form of the two-dimensional Toda lattice is derived.

Before going into details, let us review one known result on the two-dimensional Toda lattice (1). It is known that the two-dimensional Toda lattice (1) has the following solutions expressed by Casorati determinant [2]:

$$\tau_n = \begin{vmatrix} \phi_1(n) & \phi_1(n+1) & \cdots & \phi_1(n+N-1) \\ \phi_2(n) & \phi_2(n+1) & \cdots & \phi_2(n+N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N(n) & \phi_N(n+1) & \cdots & \phi_N(n+N-1) \end{vmatrix}, \tag{2}$$

where $\phi_i(m)$ satisfies the following relations:

$$\frac{\partial \phi_i(m)}{\partial x} = \phi_i(m+1), \tag{3}$$

$$\frac{\partial \phi_i(m)}{\partial s} = -\phi_i(m-1). \tag{4}$$

In order to Pfaffianize the two-dimensional Toda lattice, we require a Pfaffian with elements satisfying the following relations:

$$\frac{\partial}{\partial x}\,\mathrm{pf}(i,j)_n = \mathrm{pf}(i+1,j)_n + \mathrm{pf}(i,j+1)_n, \tag{5}$$

$$\frac{\partial}{\partial s}\,\mathrm{pf}(i,j)_n = -\,\mathrm{pf}(i-1,j)_n - \mathrm{pf}(i,j-1)_n, \tag{6}$$

$$\mathrm{pf}(i,j)_{n+1} = \mathrm{pf}(i+1,j+1)_n. \tag{7}$$

It seems reasonable for us to call (5)–(7) the Pfaffianized form of the dispersion relations (3) and (4) as (5)–(7) may be derived by means of (3) and (4) in the case of Wronski or Casorati-type Pfaffian. Thus if we take

$$f_n = \mathrm{pf}(1,2,\ldots,N)_n$$

with N being even, then we can calculate f_{n+1}, f_{n-1}, $f_{n,x}$, $f_{n,s}$ and $f_{n,xs}$ as follows:

$$f_{n+1} = \mathrm{pf}(2,3,\ldots,N,N+1), \qquad f_{n-1} = \mathrm{pf}(0,1,\ldots,N-2,N-1), \tag{8}$$

$$f_{n,x} = \mathrm{pf}(1,2,\ldots,N-1,N+1), \qquad f_{n,s} = -\,\mathrm{pf}(0,2,\ldots,N-1,N), \tag{9}$$

$$f_{n,xs} = -\,\mathrm{pf}(1,2,\ldots,N-1,N) - \mathrm{pf}(0,2,\ldots,N-1,N+1), \tag{10}$$

where we have denoted $\mathrm{pf}(1,2,\ldots,N-1,N)$ to be $\mathrm{pf}(1,2,\ldots,N-1,N)_n$ for simplicity without any confusion. Following the Hirota–Ohta's procedure, we now introduce two new variables g_n and $\hat{g}_n$,

$$g_n = \mathrm{pf}(0,1,\ldots,N,N+1)_n, \qquad \hat{g}_n = \mathrm{pf}(2,3,\ldots,N-2,N-1)_n.$$

Then we can show that f_n, g_n and $\hat{g}_n$ so defined satisfy the following three bilinear equations:

$$D_x D_s f_n \cdot f_n = 2(e^{D_n} - 1) f_n \cdot f_n - g_n \hat{g}_n, \tag{11}$$
$$D_x e^{-1/2D_n} g_n \cdot f_n = -D_s e^{1/2D_n} g_n \cdot f_n, \tag{12}$$
$$D_x e^{-1/2D_n} f_n \cdot \hat{g}_n = -D_s e^{1/2D_n} f_n \cdot \hat{g}_n. \tag{13}$$

In fact, substitution of (8)–(10) into (11) leads to the following Pfaffian identity [1,2]:

$$\begin{aligned}
&\mathrm{pf}(a_1, a_2, \ldots, a_{N-2}, \alpha, \beta, \gamma, \delta)\,\mathrm{pf}(a_1, a_2, \ldots, a_{N-2}) \\
&\quad - \mathrm{pf}(a_1, a_2, \ldots, a_{N-2}, \alpha, \beta)\,\mathrm{pf}(a_1, a_2, \ldots, a_{N-2}, \gamma, \delta) \\
&\quad + \mathrm{pf}(a_1, a_2, \ldots, a_{N-2}, \alpha, \gamma)\,\mathrm{pf}(a_1, a_2, \ldots, a_{N-2}, \beta, \delta) \\
&\quad - \mathrm{pf}(a_1, a_2, \ldots, a_{N-2}, \alpha, \delta)\,\mathrm{pf}(a_1, a_2, \ldots, a_{N-2}, \beta, \gamma) = 0,
\end{aligned} \tag{14}$$

where the list $\{a_1, a_2, \ldots, a_{N-2}\}$ is given by $\{2, 3, \ldots, N-1\}$ and the list $\{\alpha, \beta, \gamma, \delta\}$ is chosen to be $\{0, 1, N, N+1\}$. Therefore Eq. (11) holds. On the other hand, we have

$$f_{n+1,x} = \mathrm{pf}(2, 3, \ldots, N, N+2), \qquad g_{n+1} = \mathrm{pf}(1, 2, \ldots, N+1, N+2), \tag{15}$$
$$g_{n,x} = \mathrm{pf}(0, 1, \ldots, N, N+2), \qquad g_{n+1,s} = -\mathrm{pf}(0, 2, \ldots, N+1, N+2). \tag{16}$$

Substituting (15) and (16) into an equivalent form of Eq. (12),

$$D_x g_n \cdot f_{n+1} = -D_s g_{n+1} \cdot f_n, \tag{17}$$

we can easily show that Eq. (17) is reduced to the following Pfaffian identity [1,2]:

$$\begin{aligned}
&\mathrm{pf}(a_1, a_2, \ldots, a_{N-1}, \alpha, \beta, \gamma)\,\mathrm{pf}(a_1, a_2, \ldots, a_{N-1}, \delta) \\
&\quad - \mathrm{pf}(a_1, a_2, \ldots, a_{N-1}, \alpha, \beta, \delta)\,\mathrm{pf}(a_1, a_2, \ldots, a_{N-1}, \gamma) \\
&\quad + \mathrm{pf}(a_1, a_2, \ldots, a_{N-1}, \alpha, \gamma, \delta)\,\mathrm{pf}(a_1, a_2, \ldots, a_{N-1}, \beta) \\
&\quad - \mathrm{pf}(a_1, a_2, \ldots, a_{N-1}, \beta, \gamma, \delta)\,\mathrm{pf}(a_1, a_2, \ldots, a_{N-1}, \alpha) = 0,
\end{aligned} \tag{18}$$

where the list $\{a_1, a_2, \ldots, a_{N-1}\}$ represents $\{2, 3, \ldots, N\}$ and the list $\{\alpha, \beta, \gamma, \delta\}$ is chosen to be $\{0, 1, N+1, N+2\}$. Thus, (12) holds. Similarly, we can prove that Eq. (13) also holds. Therefore Eqs. (11)–(13) constitute Pfaffianized two-dimensional Toda lattice. If we wish to consider solutions to this Pfaffianized two-dimensional Toda lattice (11)–(13), then we may choose entries in the Pfaffian to be expressed in the form

$$\mathrm{pf}(i, j)_n = \sum_{k=1}^{M} \big[\Phi_k(n+i)\Psi_k(n+j) - \Phi_k(n+j)\Psi_k(n+i)\big],$$

where $\Phi_k(m)$ and $\Psi_k(m)$ satisfy

$$\frac{\partial}{\partial x}\Phi_k(m) = \Phi_k(m+1), \qquad \frac{\partial}{\partial x}\Psi_k(m) = \Psi_k(m+1), \tag{19}$$
$$\frac{\partial}{\partial s}\Phi_k(m) = -\Phi_k(m-1), \qquad \frac{\partial}{\partial s}\Psi_k(m) = -\Psi_k(m-1). \tag{20}$$

By the dependent variable transformation

$$u(n) = \ln f_n, \quad v(n) = g_n/f_n, \quad w(n) = \hat{g}_n/f_n,$$

we may transform the Pfaffianized two-dimensional Toda lattice (11)–(13) into

$$u_{xs}(n) = e^{u(n+1)+u(n-1)-2u(n)} - 1 - \frac{1}{2}v(n)w(n), \tag{21}$$

$$\begin{aligned} &v_x(n) + v_s(n+1) + v(n+1)\big(u_s(n+1) - u_s(n)\big) \\ &\quad - v(n)\big(u_x(n+1) - u_x(n)\big) = 0, \end{aligned} \tag{22}$$

$$\begin{aligned} &w_s(n) + w_x(n+1) + w(n+1)\big(u_x(n+1) - u_x(n)\big) \\ &\quad - w(n)\big(u_s(n+1) - u_s(n)\big) = 0. \end{aligned} \tag{23}$$

To summarize, in this paper, Hirota–Ohta's Pfaffianization procedure has been utilized to generate a Pfaffianized two-dimensional Toda lattice. Soliton solutions expressed by Pfaffians to the Pfaffianized two-dimensional Toda lattice have been obtained. Since the Pfaffianized two-dimensional Toda equations (11)–(13) are a coupled extension of the two-dimensional Toda lattice, it is quite natural for us to see if the Pfaffianized two-dimensional Toda lattice possesses some other integrable properties shared by the two-dimensional Toda lattice (1). As we know, one of integrable properties shared by higher dimensional integrable equations such as the two-dimensional Toda lattice (1) is existence of Lie symmetries with some arbitrary functions [9–20]. Let us now take a look at the Pfaffianized two-dimensional Toda equations (11)–(13). From the definition of a symmetry given in [21,22] , we find that $(\sigma_n, \tau_n, \rho_n)^T$ is a symmetry of the system (11)–(13) if σ_n, τ_n, ρ_n satisfy the following symmetry equations:

$$D_x D_s \sigma_n \cdot f_n = 2\big(\cosh(D_n) - 1\big)\sigma_n \cdot f_n - \frac{1}{2}(\tau_n \hat{g}_n + g_n \rho_n), \tag{24}$$

$$D_x e^{-1/2D_n}(\tau_n \cdot f_n + g_n \cdot \sigma_n) = -D_s e^{1/2D_n}(\tau_n \cdot f_n + g_n \cdot \sigma_n), \tag{25}$$

$$D_x e^{-1/2D_n}(\sigma_n \cdot \hat{g}_n + f_n \cdot \rho_n) = -D_s e^{1/2D_n}(\sigma_n \cdot \hat{g}_n + f_n \cdot \rho_n). \tag{26}$$

By some calculations, we have found the following Lie symmetries for the Pfaffianized two-dimensional Toda equation (11)–(13):

$$\begin{aligned} \sigma_n^{(1)} &= \big(h_1(x) + h_2(s)\big) f_n, \\ \tau_n^{(1)} &= \big(h_1(x) + h_2(s)\big) g_n, \\ \rho_n^{(1)} &= \big(h_1(x) + h_2(s)\big) \hat{g}_n; \end{aligned}$$

$$\begin{aligned} \sigma_n^{(2)} &= \big(h_1(x) + h_2(s)\big) n f_n, \\ \tau_n^{(2)} &= \big(h_1(x) + h_2(s)\big)(n+1) g_n, \\ \rho_n^{(2)} &= \big(h_1(x) + h_2(s)\big)(n-1) \hat{g}_n; \end{aligned}$$

$$\begin{aligned} \sigma_n^{(3)} &= h_1(x) f_{n,x} + \frac{1}{2}n^2 \dot{h}_1(x) f_n + s h_1(x) f_n, \\ \tau_n^{(3)} &= h_1(x) g_{n,x} + \frac{1}{2}(n+1)^2 \dot{h}_1(x) g_n + s h_1(x) g_n, \\ \rho_n^{(3)} &= h_1(x) \hat{g}_{n,x} + \frac{1}{2}(n-1)^2 \dot{h}_1(x) \hat{g}_n + s h_1(x) \hat{g}_n; \end{aligned}$$

$$\sigma_n^{(4)} = h_2(s) f_{n,s} + \frac{1}{2} n^2 \dot{h}_2(s) f_n + x h_2(s) f_n,$$
$$\tau_n^{(4)} = h_2(s) g_{n,s} + \frac{1}{2} (n-1)^2 \dot{h}_2(s) g_n + x h_2(s) g_n,$$
$$\rho_n^{(4)} = h_2(s) \hat{g}_{n,s} + \frac{1}{2} (n+1)^2 \dot{h}_2(s) \hat{g}_n + x h_2(s) \hat{g}_n,$$

where $h_1(x)$ and $h_2(s)$ are two arbitrary functions of x and s, respectively, and

$$\dot{h}_i(t) = \frac{d}{dt} h_i(t).$$

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